UDC 62-50

COMPUTATION OF VALUE FOR CERTAIN DIFFERENTIAL GAMES

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Two types of nonlinear differential games with fixed instant of ending are considered. Formulas are derived for value functions under particular conditions.

1. Let us consider a system of two controllable objects defined by the equations

$$x^{\cdot} = A\,(t)\,x + u, \quad x \in R^n, \quad u\,(t) \in P\,(t) \tag{1.1}$$

$$y^{\cdot} = g\,(t, y, v), \quad y \in R^m, \quad v\,(t) \in Q \qquad \text{where} \tag{1.2}$$

x and y are phase vectors of the objects; $A\,(t)$ is an n-dimensional matrix continuously dependent on t; the control vectors u, and v are bounded by the compacta $P\,(t)$ and Q, with the pointwise-multiple mapping of $P\,(t)$ bounded and measurable. The continuous function $R\,(x, y)$ defines the payoff. The player who controls object x strives to minimize the quantity $R\,(x\,(\theta), y\,(\theta))$ which represents the payoff in the phase vector system at instant $t = \theta$ of the game end, while the player controlling object y strives to maximize the payoff.

It is assumed that the conditions which ensure the existence and uniqueness of solution of Eq. (1.2) up to $t = \theta$ for any initial conditions and any measurable function $v\,(t) \in Q$ are satisfied. These conditions are: function $g\,(t, y, v)$ must be continuous over the totality of its arguments and must satisfy the local Lipschitz condition for y uniformly with respect to v. It is assumed that $\|\, g\,(t, y, v)\,\| \leqslant \varkappa\,(1 + \|\, y\,\|)$ and $\varkappa = \text{const}$.

We denote the phase vector and the space of the system by $z = (x, y)$ and $R^k = R^n \times R^m$, and use the concepts of the theory of differential games formulated in book [1].

According to [1] a value function $\varepsilon\,(t, z)$ exists for the game considered here. We shall seek its form for the position (t_0, z_0), assuming that the payoff $R\,(z)$ can be represented in the form

$$R\,(z) = \max_{s \in S} R_s\,(z), \quad R_s\,(z) = R_s^{\,1}\,(x) + R_s^{\,2}\,(y) \tag{1.3}$$

where S is a compactum and function $R_s\,(z)$ is continuous with respect to (s, z). We denote by $\varepsilon_s^{\,1}\,(t_*, x_*)$ the "value" of the following problem of optimal control:

$$x^{\cdot} = A\,(t)\,x + u, \quad x\,(t_*) = x_*, \quad u\,(t) \in P\,(t); \quad R_s^{\,1}\,(x\,(\theta)) \to \inf$$

Similarly, the quantity $\varepsilon_s^{\,2}\,(t_*, y_*)$ relates to the problem

$$y^{\cdot} = g\,(t, y, v), \quad y\,(t_*) = y_*, \quad v\,(t) \in Q$$
$$R_s^{\,2}\,(y\,(\theta)) \to \sup$$

Let us consider function

$$\varepsilon^*(t, z) = \max_{s \in S} \{\varepsilon_s^1(t, x) + \varepsilon_s^2(t, y)\} \tag{1.4}$$

which is continuous and has a maximum because function $\varepsilon_s(t, z) = \varepsilon_s^1(t, x) + \varepsilon_s^2(t, y)$ is continuously dependent on (s, t, z).

Let us define the sufficient conditions for the equality $\varepsilon^*(t_0, z_0) = \varepsilon(t_0, z_0)$ to be satisfied. We introduce for any c the closed sets

$$W_c(t) = \{z \in R^k : \varepsilon(t, z) \leqslant c\}$$
$$W_c^*(t) = \{z \in R^k : \varepsilon^*(t, z) \leqslant c\}$$
$$W_c^s(t) = \{z \in R^k : \varepsilon_s(t, z) \leqslant c\}$$

and use symbol ∂ for denoting the boundary of the set in R^k.

Condition 1.1. Let $\varepsilon^*(t_0, z_0) = c_0$. Then

$$W_{c_0}^*(t) \neq \varnothing, \quad \forall t \in [t_0, \theta]$$

If condition 1.1 is satisfied, there exists a collection of closed convex sets $B(t) \subset R^k$ which depend on $t \in [t_0, \theta]$ and such that:

1) $z_0 \in B(t_0)$;

2) set $B(t_2)$ contains for any $t_0 \leqslant t_1 < t_2 \leqslant \theta$ all phase positions that can be reached at instant t_2 from position (t_1, z_1), where $z_1 \in B(t_1)$, and

3) $W_{c_0}^*(t) \cap B(t) \neq \varnothing, \quad \forall t \in [t_0, \theta]$.

Let there exist an open convex set $B \subset R^k$, which contains set $B(t)$ with properties defined above, and such that the following conditions are satisfied.

Condition 1.2. Function $\varepsilon_s(t, z)$ must be convex over set B relative to z for any $s \in S$ and $t \in [t_0, \theta]$. Note that this condition implies the convexity of sets $W_c^*(t) \cap B$.

Condition 1.3. If the number $\beta > 0$ is such that for every $c \in (c_0, c_0 + \beta)$ there exists a set J_c which is dense in $[t_0, \theta]$ and has the following properties. The part of the boundary of set $W_c^*(t)$ in B is smooth for any $t \in J_c$, i.e. it is possible to draw from every point in $\partial W_c^*(t) \cap B$ a unique supporting hyperplane to $W_c^*(t) \cap B$.

Theorem 1. If conditions 1.1 - 1.3 are satisfied, $\varepsilon^*(t_0, z_0) = \varepsilon(t_0, z_0)$.

Proof. We denote by $T[t_1, t_2]\{M\}$ the set of program absorption [1], i.e. the set of all points $z_1 \in R^k$, such that the first player is able to bring the system from position (t_1, z_1) to position (t_2, z_2) for any arbitrary $t_1 < t_2$ from $[t_0, \theta]$ and the set $M \subset R^k$, if he knows the programed control of the second player in the interval $[t_1, t_2]$. At the position (t_2, z_2), $z_2 \in M$.

Evidently $\varepsilon^*(t_0, z_0) \leqslant \varepsilon(t_0, z_0)$, hence for proving the theorem it is sufficient to show that the inclusion

$$z_0 \in W_c(t_0) \tag{1.5}$$

is valid for any $c \in (c_0, c_0 + \beta)$. Let us prove (1.5) for some fixed c, noting that property 3) of set $B(t)$ implies that $W_c^*(t) \cap B(t) \neq \varnothing$ for all $t \in [t_0, \theta]$.

Suppose that the following statement has been already proved. Inclusion

$$T\,[\tau_1, \tau_2]\,\{W_c^*(\tau_2) \cap B(\tau_2)\} \supset W_c^*(\tau_1) \cap B(\tau_1) \tag{1.6}$$

is valid for any τ_1 and τ_2 such that $\tau_2 \in J_c$ and $t_0 \leqslant \tau_1 < \tau_2$. The validity of (1.5) follows from this statement.

Let us consider the subdivision of segment $[t_0, \theta]$ by points $t_0 < t_1 < \cdots < t_N < \theta$ such that $t_i \in J_c$ when $1 \leqslant i \leqslant N$. Then from (1.6) we have

$$T\,[t_0, t_1] \ldots T\,[t_{N-1}, t_N]\,\{W_c^*(t_N) \cap B(t_N)\} \supset W_c^*(t_0) \cap B(t_0) \tag{1.7}$$

By reducing the size of subdivisions of segment $[t_0, t_N]$ by points from J_c and using the differential game lattice [1,2], from (1.7) we obtain

$$S\,[t_0, t_N]\,\{W_c^*(t_N) \cap B(t_N)\} \supset W_c^*(t_0) \cap B(t_0) \tag{1.8}$$

where $S\,[a, b]\,\{M\}$ denotes the set of points $z \in R^k$ such that position (a, z) is the point of local absorption of set $M \subset R^k$ at instant $t = b$ [1].

Since function $\varepsilon^*(t, z)$ is continuous with respect to (t, z), the set $W_c^*(t)$ is upper semicontinuous with respect to t. Since owing to property 2) set $B(t)$ is upper semicontinuous on the left, hence set $W_c^*(t) \cap B(t)$ is also upper semicontinuous on the left. From this and the theorem on alternative [1] we can deduct that set $S\,[t_0, t]\,\{W_c^*(t) \cap B(t)\}$ is also upper semicontinuous on the left with respect to t. Hence from (1.8) taking into account that $W_c^*(\theta) = W_c(\theta)$ and $W_c(t_0) = S\,[t_0, \theta]\,\{W_c(\theta)\}$ we obtain

$$\begin{aligned} &W_c^*(t_0) \cap B(t_0) \subset S\,[t_0, \theta]\,\{W_c^*(\theta) \cap B(\theta)\} \subset \\ &S\,[t_0, \theta]\,\{W_c^*(\theta)\} = W_c(t_0) \end{aligned} \tag{1.9}$$

Since $z_0 \in W_c^*(t_0) \cap B(t_0)$, from (1.9) follows (1.5).

It remains to verify the statement (1.6). For this it is sufficient to prove the equality

$$T\,\{E\} = \bigcap_{s \in S} T\,\{E_s\} \tag{1.10}$$

where $T = T\,[\tau_1, \tau_2]$, τ_1 and τ_2 are fixed and satisfy the assumptions of statement (1.6), and $E = W_c^*(\tau_2) \cap B(\tau_2)$ and $E_s = W_c^s(\tau_2) \cap B(\tau_2)$).

Since function $\varepsilon_s(t, z)$ represents the game value and, also, the program maximin for system (1.1), (1.2) and for the payoff $R_s(z)$ [1], hence

$$T\,\{W_c^s(\tau_2)\} = W_c^s(\tau_1), \quad \forall s \in S$$

and owing to property 2) of set $B(t)$ we have

$$T\,\{E_s\} \supset W_c^s(\tau_1) \cap B(\tau_1) \tag{1.11}$$

Using (1.10) and (1.11) we obtain (1.6) in the form

$$T\,\{E\} = \bigcap_{s \in S} T\,\{E_s\} \supset \bigcap_{s \in S} W_c^s(\tau_1) \cap B(\tau_1) = W_c^*(\tau_1) \cap B(\tau_1)$$

First, let us consider the case when E is a compactum. We represent E as the intersection of supporting half-planes

$$E = \bigcap_{l \in \partial D} O_l, \quad O_l = \{z \in R^k : \langle l, z\rangle \leqslant \max_{q \in E} \langle l, q\rangle\}$$

where D is the unit sphere in R^k. We shall prove that for any $l_* \in \partial D$ there exists an element $s_* \in S$ such that

$$O_{l_*} \supset E_{s_*} \qquad (1.12)$$

Let $z_* \in \partial E$ be a point such that the hyperplane $\Pi(l_*) = \{z \in R^k : \langle l_*, z\rangle = \langle l_*, z_*\rangle\}$ represents the support of E. Since $z_* \in \partial W_c^*(\tau_2) \cup \partial B(\tau_2)$, three cases are possible.

1°. $z_* \in \partial B(\tau_2)$ and $z_* \notin \partial W_c^*(\tau_2)$, when (1.12) is evidently satisfied for any $s_* \in S$.

2°. $z_* \notin \partial B(\tau_2)$ and $z_* \in \partial W_c^*(\tau_2)$. Since $\varepsilon^*(\tau_2, z_*) = c$, there exists an element $s_* \in S$ such that $\varepsilon_{s_*}(\tau_2, z_*) = c$. We shall prove that $z_* \in \partial W_c^{s_*}(\tau_2)$. If $z_* \in \operatorname{int} W_c^{s_*}(\tau_2)$, then, owing to the convexity of function $\varepsilon_{s_*}(\tau_2, z)$ with respect to $z \in B$, we would have $c = \min\{\varepsilon_{s_*}(\tau_2, z) : z \in B(\tau_2)\}$. However, since $t_0 < \tau_2$, then $c \leqslant \inf\{\varepsilon_{s_*}(t_0, z) : z \in B(t_0)\}$ and, consequently, $c \leqslant \inf\{\varepsilon^*(t_0, z) : z \in B(t_0)\}$, which contradicts the inequality $c > c_0 = \varepsilon^*(t_0, z_0)$. Hence $z_* \in \partial W_c^{s_*}(\tau_2)$. Since $z_* \notin \partial B(\tau_2)$, the hyperplane $\Pi(l_*)$ is a supporting one and because of condition 1.3 it is, also, the unique support for $W_c^*(\tau_2) \cap B$ that passes through point z_*. Since $W_c^*(\tau_2) \subset W_c^{s_*}(\tau_2)$, any hyperplane that passes through point z_* and is a supporting one for set $W_c^{s_*}(\tau_2) \cap B$, is also supporting for $W_c^*(\tau_2) \cap B$. These two observations imply that the hyperplane $\Pi(l_*)$ is a supporting one for $W_c^{s_*}(\tau_2) \cap B$ and, consequently, also for E_{s_*}. Hence (1.12) is also valid in case 2°.

3°. $z_* \in \partial W_c^*(\tau_2) \cap \partial B(\tau_2)$. We assume that the hyperplane $\Pi(l_*)$ is not a supporting plane for $W_c^*(\tau_2) \cap B$, as otherwise the previous reasoning could be applied. As in case 2°, we assume that $s_* \in S$. If $\Pi(l_*)$ is not a supporting hyperplane for set $W_c^{s_*}(\tau_2) \cap B(\tau_2)$, points $z_1 \in W_c^{s_*}(\tau_2) \cap B(\tau_2)$ and $z_2 \in W_c^{s_*}(\tau_2) \cap (B \setminus B(\tau_2))$ would be found lying outside the half-space O_{l_*}. The existence of such points z_1, z_2, and z_* contradicts the convexity of set E_{s_*}.

Thus the statement (1.12) is valid in all three cases, and implies that

$$\bigcap_{l \in \partial D} T\{O_l\} \supset \bigcap_{s \in S} T\{E_s\} \qquad (1.13)$$

But by Neumann's minimax theorem we have for system (1.1), (1.2)

$$T\{\bigcap_{l \in \partial D} O_l\} = \bigcap_{l \in \partial D} T\{O_l\} \qquad (1.14)$$

From (1.13) and (1.14) we obtain $T\{E\} \supseteq \bigcap_{s \in S} T\{E_s\}$. The inverse inclusion is obvious. Hence (1.10) is proved in the case when E is a compactum.

When set E is unbounded, the proof is reduced to the previous one by the following procedure.

To prove (1.10) it is sufficient to show that

$$rD \cap T\{E\} = \bigcap_{s \in S} [rD \cap T\{E_s\}], \quad \forall r > 0$$

We set $r = r_0$. A reasonably large number r_1 can then be found such that when sets E° and E_s° satisfy the relationship

$$r_1 D \cap E = r_1 D \cap E^\circ, \quad r_1 D \cap E_s = r_1 D \cap E_s^\circ, \quad \forall s \in S$$

then

$$r_0D \cap T\{E\} = r_0D \cap T\{E^\circ\}, \quad r_0D \cap T\{E_s\} = r_0D \cap T\{E_s^\circ\}, \quad \forall s \in S$$

We set $E^\circ = E \cap r_1D$ and $E_s^\circ = E_s \cap r_1D$. Since E° is a compactum, hence, as previously shown, we have

$$T\{E^\circ\} = \bigcap_{s\in S} T\{E_s^\circ\}$$

This proves (1.10) and completes the proof of the theorem.

Note. If Eq. (1.2) is linear and the payoff $R(z)$ is the Euclidean distance to the convex compactum in R^k, then mapping (1.3) contains linear $R_s^1(x)$ and $R_s^2(y)$, and function $\varepsilon^*(t, z)$ coincides with the programed maximin. Condition (1.2) is satisfied for $B = R^k$.

Condition 1.3 is satisfied in the case of a regular problem [1]. It should be noted that when the dependence of $P(t)$ on t is continuous, the differentiability of function $\varepsilon^*(t, z)$ not only with respect to z but, also, to t follows from the condition of regularity. This implies that $\varepsilon^{*\cdot} = \varepsilon$, which means that condition (1.1) is also satisfied. If, however, the dependence of $P(t)$ on t is measurable but discontinuous, condition 1.1 may not be satisfied, and has to be postulated.

Example. Let us consider the modification of the problem in [3]. Let system (1.1), (1.2) be presented in the form

$$x_1^\cdot = x_2, \quad x_2^\cdot = u_1, \quad x_3^\cdot = x_4, \quad x_4^\cdot = u_2; \quad u(t) \in P(t)$$
$$y_1^\cdot = y_2, \quad y_2^\cdot = \lambda y_2^2 + v_1, \quad y_3^\cdot = y_4, \quad y_4^\cdot = v_2; \quad v(t) \in Q$$
$$Q = \{v = (v_1, v_2) : \|v\| \leqslant \nu\}, \quad P(t) = \{u = (u_1, u_2) : \|u\| \leqslant \mu(t)\}$$

where $\mu(t)$ is a measurable bounded positive function, and the number $\lambda > 0$ is a small parameter.

Let

$$R(x, y) = \sqrt{(y_1 - x_1)^2 + (y_3 - x_3)^2} + a_1(y_1 - x_1) + a_3(y_3 - x_3)$$

where a_1 and a_3 are numbers. The game is considered in the time interval $[0, \theta]$. The payoff $R(x, y)$ can be represented in the form (1.3), i.e.

$$R(x, y) = \max_{s\in S} \{(s_1y_1 + s_3y_3) - (s_1x_1 + s_3x_3)\}$$
$$S = \{s = (s_1, s_3): \quad (s_1 - a_1)^2 + (s_3 - a_3)^2 \leqslant 1\}$$

We assume that $a_1 > 1$ and $\mu(t) - \nu \geqslant \alpha > 0$ for all t. It was shown in [3] that

$$\varepsilon_s(t, x, y; \lambda) = -k(t)\|s\| + s_1((y_1 - x_1) + (\theta - t)(y_2 - x_2)) + s_3((y_3 - x_3) + (\theta - t)(y_4 - x_4)) + {}^1/_6\,\lambda s_1(\theta - t)^2\{3y_2^2 + 2\nu y_2(s_1/\|s\| - s_3^2/\|s\|^2)(\theta - t) - \nu^2(\theta - t)^2 s_1({}^5/_2\, s_1/\|s\|^2 + s_3^2/\|s\|^3)\} + \lambda^2(\theta - t)^2 f(t, \lambda, s, y_2)$$

$$k(t) = \int_t^\theta (\theta - \tau)(\mu(\tau) - \nu)\,d\tau$$

Note that function $f(t, \lambda, s, y_2)$ is positive homogeneous with respect to s, and that its second derivatives with respect to $s \in S$ and the second derivative with respect to y_2 continuously depend on t, λ, s, and y_2 in the region of their variation.

Let us take compactum Γ in space (t, x, y) and show that when λ_* is fairly small we have $\varepsilon^* (t_0, x_0, y_0; \lambda_0) = \varepsilon (t_0, x_0, y_0; \lambda_0)$ for any $\lambda_0 \leqslant \lambda_*$ and $(t_0, x_0, y_0) \in \Gamma$.

Since circle S does not contain 0, it is possible to find a sphere $B_* \subset R^k$ of radius r_* with its center at zero, such that for any $\lambda_0 \leqslant 1$ and $(t_0, x_0, y_0) \in \Gamma$ there exists set $B_{(\lambda_0, t_0, x_0, y_0)} (t)$ with properties 1) - 3) and is contained in B_*. The subscript at $B (t)$ indicates that the set is chosen for the initial position (t_0, x_0, y_0) and parameter $\lambda = \lambda_0$).

We shall show that there exists a $\lambda_* \leqslant 1$ such that function $\varepsilon_s (t, x, y; \lambda)$ is convex relative to (x, y) and concave relative to s, when $t \in [0, \theta]$, $(x, y) \in B_*$, $s \in S$ and $\lambda \leqslant \lambda_*$.

Since for every $s = (s_1, s_3) \in S$ $s_1 \geqslant \delta > 0$ (δ is some number), function $^1/_2\, \lambda s_1 y_2^2 + \lambda^2 f (t, \lambda, s, y_2)$ is convex relative to y_2 in the set $| y_2 | \leqslant r_*$ for all $t \in [0, \theta]$ and $s \in S$, if $\lambda \leqslant \lambda_1$ (λ_1 is fairly small). This implies convexity of function $\varepsilon_s (t, x, y; \lambda)$ relative to $(x, y) \in B_*$.

Since $k (t) \geqslant {}^1/_2 \alpha (\theta - t)^2$, hence function $\varepsilon_s (t, x, y; \lambda)$ is concave relative to $s \in S$ when $\lambda \leqslant \lambda_2$ (λ_2 is fairly small). We set $\lambda_* = \min (\lambda_1, \lambda_2)$.

We take arbitrary $\lambda_0 \leqslant \lambda_*$ and $(t_0, x_0, y_0) \in \Gamma$, and shall check if conditions 1. 2 and 1. 3 are satisfied. For set B we take the sphere B_*. Condition 1. 2 is then satisfied. We set $\varepsilon^* (t_0, x_0, y_0; \lambda_0) = c_0$, select $\beta > 0$ so that $0 \notin (c_0, c_0 + \beta)$, and assume that $c \in (c_0, c_0 + \beta)$. We then check if the part of the boundary of set $W_c^* (t)$, located in B_*, is smooth for any $t \in [t_0, \theta]$. For this it is sufficient to show that for any $t \in [t_0, \theta]$, $(x, y) \in B_*$ for which $\varepsilon^* (t, x, y; \lambda_0) = c$ the maximum in the equality $\varepsilon^* (t, x, y, \lambda_0) = \max \{\varepsilon_s (t, x, y; \lambda_0) : s \in S\}$ is reached on a unique s. This follows from the condition $\varepsilon^* (t, x, y; \lambda_0) = c \neq 0$ of positive homogeneity and concavity of function $\varepsilon_s (t, x, y; \lambda_0)$ relative to $s \in S$. This proves that conditions 1. 1 - 1. 3 are satisfied for the considered here λ_0 and (t_0, x_0, y_0) . Hence Theorem 1 is valid.

Note that when function $\mu (t)$ is continuous, the equality $\varepsilon^* (t_0, x_0, y_0; \lambda_0) = \varepsilon (t_0, x_0, y_0; \lambda_0)$ may be solved more simply by the method used in [3].

2 . Let us consider the differential game ending at instant $t = \theta$. The motion of the system is specified by the linear equation

$$x^{\cdot} = u + v, \quad x \in R^n, \quad u (t) \in P (t), \quad v (t) \in Q (t) \tag{2. 1}$$

where x is the system phase vector and the dependence of compacta $P (t)$ and $Q (t)$ in R^n on t is measurable and bounded. Let the continuous payoff function $\Gamma (x)$ be of the form

$$\Gamma (x) = \min_{s \in S} \max_{l \in L} \gamma (x; s, l) \tag{2. 2}$$
$$\gamma (x; s, l) = \langle a (s, l), x \rangle + b (s, l)$$

where S and L are convex compacta, function $\gamma (x; s, l)$ is convex relative to $s \in S$, concave relative to $l \in L$, and affine relative to x ; $a (s, l)$ is a continuous function with values in R^n, and the scalar function $b (s, l)$ is lower semicontinuous with respect to s and upper semicontinuous with respect to l.

Representation (2. 2) is admissible in the following cases:

a) $\Gamma (x) = \min \{\Lambda_1 (x), \ldots, \Lambda_k (x), \varphi (x)\}$ where $\Lambda_i (x)$ are linear functions and $\varphi (x)$ is a convex function such that dom φ^* is a compactum (see [4]), and

b) $\Gamma(x) = \varphi_1(x) - \varphi_2(x)$, where the convex functions $\varphi_i(x)$ are such that the sets $\operatorname{dom} \varphi_i^*$ are compacta.

We introduce the notation

$$\varkappa(t, x; s, l) = \langle a(s, l), x\rangle + \int_t^\theta \min_{u \in P(\tau)} \langle a(s, l), u\rangle \, d\tau + \int_t^\theta \max_{v \in Q(\tau)} \langle a(s, l), v\rangle \, d\tau + b(s, l)$$

$$\varepsilon_{00}(t, x) = \max_{l \in L} \min_{s \in S} \varkappa(t, x; s, l), \quad \varepsilon^{\infty}(t, x) = \min_{s \in S} \max_{l \in L} \varkappa(t, x; s, l)$$

and denote the value function by $\varepsilon(t, x)$.

We assume that the following condition is satisfied.

Condition 2.1. Function $\varkappa(t, x; s, l)$ must be convex relative to s and concave relative to l in the set $S \times L$ for any position (t, x).

Theorem 2. If condition 2.1 is satisfied, then for all positions

$$\varepsilon_{00}(t, x) = \varepsilon(t, x) = \varepsilon^{\infty}(t, x)$$

Proof. First, we would point out that according to one extension of Neumann's theorem on minimax [5] equality (2.2) may be represented in the form

$$\Gamma(x) = \max_{l \in L} \min_{s \in S} \gamma(x; s, l) \tag{2.3}$$

Let us prove the validity of inequality

$$\varepsilon_{00}(t, x) \leqslant \varepsilon(t, x) \tag{2.4}$$

For every $l \in L$ we introduce the continuous function of x

$$\gamma_l(x) = \min_{s \in S} \gamma(x; s, l)$$

Let $\varepsilon(t, x \mid \gamma_l(\cdot))$ be the value function of the game which corresponds to system (2.1) and to payoff function $\gamma_l(x(\theta))$. We shall prove that

$$\varepsilon(t, x \mid \gamma_l(\cdot)) = \min_{s \in S} \varkappa(t, x; s, l) \tag{2.5}$$

We apply the method used in [2], and denote by $\varepsilon^{\circ}(t_1, x_* \mid t_2, \varphi(\cdot))$ the program minimax in the game defined by system (2.1) with payoff $\varphi(x(t_2))$ for position (t_1, x_*) and any arbitrary instant of time $t_1 < t_2 \leqslant \theta$ and function $\varphi(x)$. The minimax is determined by formula

$$\varepsilon^{\circ}(t_1, x_* \mid t_2, \varphi(\cdot)) = \inf_{u(\cdot) \in U} \sup_{v(\cdot) \in V} \varphi(x[t_2; t_1, x_*, u, v])$$

where U is the set of programed controls of the first player in $[t_1, t_2]$, i.e. of measurable functions $u(\cdot)$ which satisfy almost everywhere in $[t_1, t_2]$ the constraint $u(t) \in P(t)$. The definition of set V is similar. We denote by $x[t_2; t_1, x_*, u, v]$ the system phase point at instant t_2 with initial position (t_1, x_*) and the selected controls $u(\cdot)$ and $v(\cdot)$.

It can be verified that

$$\varepsilon^\circ(t, x \mid \theta, \gamma_l(\cdot)) = \min_{s\in S} \varkappa(t, x; s, l) \tag{2.6}$$

Let us show that for any $t_1 < t_2 < \theta$ and $x_* \in R^n$

$$\varepsilon^\circ(t_1, x_* \mid t_2, \varepsilon^\circ(t_2, \cdot \mid \theta, \gamma_l(\cdot))) = \varepsilon^\circ(t_1, x_* \mid \theta, \gamma_l(\cdot)) \tag{2.7}$$

In fact, using (2.6) with allowance for the convexity of function $\varkappa(t, x; s, l)$ relative to s, we obtain

$$\varepsilon^\circ(t_1, x_* \mid t_2, \varepsilon^\circ(t_2, \cdot \mid \theta, \gamma_l(\cdot))) = \inf_{u(\cdot)\in U} \sup_{v(\cdot)\in V} \min_{s\in S} \varkappa(t_2, x[t_2; t_1, x_*, u, v]; s, l) = \inf_{s\in S} \inf_{u(\cdot)\in U} \sup_{v(\cdot)\in V} \varkappa(t_2, x[t_2; t_1, x_*, u, v]; s, l) = \min_{s\in S} \varkappa(t_1, x_*; s, l) = \varepsilon^\circ(t_1, x_* \mid \theta, \gamma_l(\cdot))$$

Equality (2.7) is proved. It implies that in virtue of the differential games lattice [1,2] $\varepsilon(t, x \mid \gamma_l(\cdot)) = \varepsilon^\circ(t, x \mid \theta, \gamma_l(\cdot))$ and, consequently, (2.5) follows from (2.6).

From (2.3) for any $l \in L$ we obtain $\gamma_l(\cdot) \leqslant \Gamma(\cdot)$, hence $\varepsilon(t, x \mid \gamma_l(\cdot)) \leqslant \varepsilon(t, x \mid \Gamma(\cdot))$. From this and equality (2.5) we obtain (2.4). Using the program maximin and equality (2.2) with allowance for the convexity of function $\varkappa(t, x; s, l)$ relative to l, we similarly obtain

$$\varepsilon(t, x) \leqslant \varepsilon^{\circ\circ}(t, x) \tag{2.8}$$

By the already mentioned theorem about the minimax we have $\varepsilon_{00}(t, x) = \varepsilon^{\circ\circ}(t, x)$, hence (2.4) and (2.8) confirm the theorem.

Example. Let system (2.1) be defined by

$$\dot{x}_1 = u_1 + v_1, \quad u(t) \in P(t) = \{u = (u_1, u_2) : \|u\| \leqslant 2(1-t)\}$$
$$\dot{x}_2 = u_2 + v_2, \quad v(t) \in Q = \{v = (v_1, v_2) : \|v\| \leqslant 1\}$$

The game is played in the time interval $[0, 1]$. The payoff is defined by $\Gamma(x) = \min\{\langle c, x\rangle, \varphi(x)\}$, where c is a nonzero vector in R^2, and the convex function $\varphi(x)$ is determined by its conjugate [4]

$$\varphi^*(l) = \begin{cases} \|l\|^2, & l \in L \\ +\infty, & l \notin L \end{cases}$$

where L is a circle of unit radius in R^2 whose center is at point d. We assume that L does not intersect the half-line directed toward vector $-c$.

Let us represent function $\Gamma(x)$ in the form (2.2)

$$\Gamma(x) = \min_{s\in S} \max_{l\in L} \{\langle s_1 c + s_2 l, x\rangle - s_2 \|l\|^2\}$$
$$S = \{s = (s_1, s_2): \quad s_1 + s_2 = 1; \quad s_1, s_2 \geqslant 0\}$$

i.e. in this example we have $a(s, l) = s_1 c + s_2 l$ и $b(s, l) = -s_2\|l\|^2$.

For any $t \in [0, 1]$ and $r \in R^2$

$$\int_t^1 \min_{u\in P(\tau)} \langle r, u\rangle d\tau + \int_t^1 \max_{v\in Q} \langle r, v\rangle d\tau = k(t)\|r\|$$

where $k(t)$ is a nonnegative function. Hence

$$\varkappa(t, x; s, l) = \langle s_1 c + s_2 l, x\rangle + k(t)\| s_1 c + s_2 l\| - s_2 \| l \|^2$$

Function $\varkappa(t, x; s, l)$ is convex relative to $s \in S$ for any $l \in L$. If the norm of vector d is fairly large, $\varkappa(t, x; s, l)$ is concave relative to $l \in L$. Hence condition 2.1 is satisfied.

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